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QCD sum rules analysis of the rare $B_c \to X\nu\bar{\nu}$ decays

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Abstract Taking into account the gluon correction contributions to the correlation function, the form factors relevant to the rare $B_c \to X\nu\bar{\nu}$ decays are calculated in the framework of the three-point QCD sum rules, where X stands for axial vector particle, $AV(D_{s1})$, and vector particles, $V(D^*, D_s^*)$. The total decay width as well as the branching ratio of these decays are evaluated using the q^2 dependent expressions of the form factors. A comparison of our results with the predictions of the relativistic constituent quark model is presented.

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1 Introduction

The discovery of the B_c meson by the CDF detector at the Fermi Lab in $p\bar{p}$ collisions via the decay mode $B_c \to J/\psi l^{\pm}\nu$ at $\sqrt{s} = 1.8$ TeV [1] has illustrated the possibility of experimental study of the charm–beauty systems and has produced considerable interest in its spectroscopy. This meson constitutes a very rich laboratory, since with the luminosity values of $\mathcal{L} = 10^{34}$ cm^{-2} s^{-1} and $\sqrt{s} = 14$TeV at LHC, the number of $B_c^{\pm}$ mesons is expected to be about 10^8–10^{10} per year [2–6]. This will provide a good opportunity to study not only some rare B_c decays, but also CP violation, T violation and polarization asymmetries. The long-lived heavy quarkonium, B_c, is the only meson containing two heavy quarks with different charge and flavors (b and c), in which its decay properties are expected to be different from those of flavor neutral mesons, and this may produce significant progress in the study of heavy quark dynamics. The B_c system is the lowest bound state of two heavy quarks with open flavor. Such states have no annihilation decay modes, due to the electromagnetic and strong interactions, since the excited levels of $\bar{b}c$ lie below the threshold of decay into a pair of heavy B and D mesons, so this meson decays weakly. Many parameters enter in the description of weak decays of this meson. In particular, measuring the branching ratios of such decays provides us with a new framework for a more precise calculation of the Cabibbo–Kobayashi–Maskawa (CKM) matrix elements V_{tq} $(q = d, s, b)$, leptonic decay constants, quark masses and mixing angles. Also, study of this meson can be used to find constraints on the physics beyond the standard model. Indeed, collection of more hadrons containing heavy quarks provides higher accuracy and more confidence in our understanding of QCD dynamics (for details of the physics of the B_c meson, see [7, 8]).

In the present work, the $B_c \to (D^*, D_s^*)\nu\bar{\nu}$ and $B_c \to D_{s1}\nu\bar{\nu}$ transitions are investigated in the framework of the three-point QCD sum rule approach. Theoretical calculation of the amplitudes for these decays is particularly reliable, owing to the absence of long-distance interactions that affect the charged-lepton channels $B_c \to Xl^+l^-$. The rare $B_c \to (D^*, D_s^*)\nu\bar{\nu}$ and $B_c \to D_{s1}\nu\bar{\nu}$ decays are proceeded by flavor changing neutral current (FCNC) transitions of $b \to s, d$. These transitions occur at loop level in the standard model (SM), and they are very sensitive to physics beyond the SM, since some new particles might have contributions in the loops diagrams. New physics such as SUSY particles or a possible fourth generation could contribute to the penguin loop or box diagram and change the branching fractions [9]. The possibility of discovering light dark matter in $b \to s$ transitions with large missing momentum has been discussed in Ref. [10]. Note that some possible B_c decays such as $B_c \to l\bar{\nu}\gamma$, $B_c \to \rho^+\gamma$, $B_c \to K^{*+}\gamma$, $B_c \to B_u^* l^+l^-$, $B_c \to B_u^*\gamma$ and $B_c \to D_{s,d}^*\gamma$ have previously been studied in the framework of light-cone and tree-point QCD sum rules [11–15]. A larger set of exclusive non-leptonic and semileptonic decays of the B_c meson, which have been

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studied within a relativistic constituent quark model, can be found in Ref. [16]. Moreover, the $B_c \to (D^*, D_s^*)\nu\bar{\nu}$ transitions have also been investigated in the framework of the relativistic constituent quark model (RCQM) [17].

This paper includes three sections. The calculation of the sum rules for the relevant form factors are presented in Sect. 2. In the sum rule expressions for the form factors, the light quark condensates do not have any contributions, so, as a first correction in the nonperturbative part of the correlator, the two gluon condensate contributions to the correlation function are taken into account. Section 3 contains a numerical analysis, discussion and a comparison of the present work results with the predictions of the RCQM.

2 Sum rules for the transition form factors of $B \to AV(V)\nu\bar{\nu}$ $[AV(V) = D_{s1}(D^*, D_s^*)]$

The $B \to X\nu\bar{\nu}$ process is described at quark level via the $b \to q\nu\bar{\nu}$ transition ($q = d$ or s) (see Fig. 1) in the SM and receives contributions from Z-penguin and box diagrams, where the dominant contributions come from an intermediate top quark. The explicit form of the effective Hamiltonian responsible for the $b \to q\nu\bar{\nu}$ decays is described by only one Wilson coefficient, namely C_{10},

$$H_{\text{eff}} = \frac{G_F \alpha}{2\pi\sqrt{2}} C_{10}\big(V_{tb}V_{tq}^*\big)\bar{q}\gamma^\mu(1-\gamma_5)b\bar{\nu}\gamma_\mu(1-\gamma_5)\nu, \tag{2.1}$$

where G_F is the Fermi constant, α is the fine structure constant at the Z mass scale, and the V_{ij} are the elements of the CKM matrix. The presence of only one operator in the effective Hamiltonian makes the $b \to q\nu\bar{\nu}$ process important, because the estimated theoretical uncertainty is related only to the value of the Wilson coefficient C_{10}. For more about the Wilson coefficients, see [13, 17–19] and references therein. The amplitudes of the $B_c \to X\nu\bar{\nu}$ decays are obtained by sandwiching equation (2.1) between the initial and final meson states:

$$\begin{aligned} M = {}& \frac{G_F \alpha}{2\pi\sqrt{2}} C_{10}\big(V_{tb}V_{tq}^*\big)\bar{\nu}\gamma_\mu(1-\gamma_5)\nu \\ & \times \big\langle X(p',\varepsilon)\big|\bar{q}\gamma_\mu(1-\gamma_5)b\big|B_c(p)\big\rangle. \end{aligned} \tag{2.2}$$

Our aim is to calculate the matrix element $\langle X(p',\varepsilon)|\bar{q}\gamma_\mu(1-\gamma_5)b|B_c(p)\rangle$ appearing in (2.2). Both the vector and axial vector part of the transition current $\bar{q}\gamma_\mu(1-\gamma_5)b$ contribute to the matrix element discussed above. Considering parity and Lorentz invariance, one can parameterize this matrix element in terms of the form factors in the following form:

$$\begin{aligned} & \big\langle V(p',\varepsilon)\big|\bar{q}\gamma_\mu b\big|B_c(p)\big\rangle \\ & \quad = -\epsilon_{\mu\nu\alpha\beta}\varepsilon^{*\nu}p^\alpha p'^\beta \frac{2V(q^2)}{m_{B_c}+m_V}, \end{aligned} \tag{2.3}$$

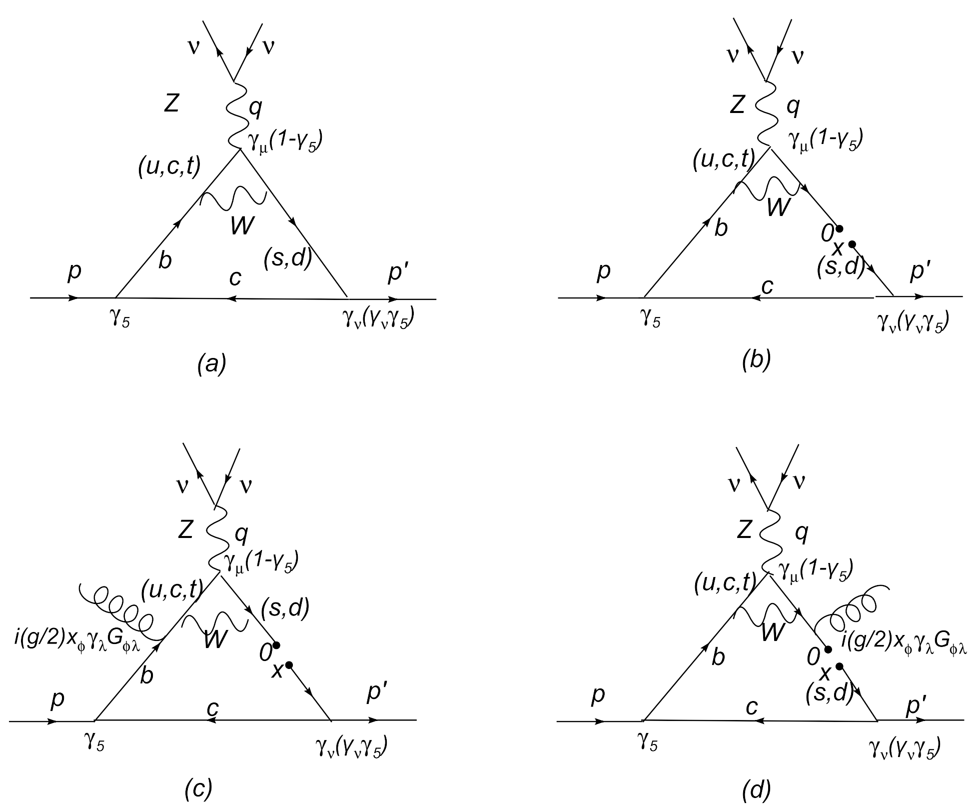

Fig. 1 Loop diagrams for $B_c \to X\nu\bar{\nu}$ transitions, bare loop (diagram **a**) and light quark condensates (without any gluon diagram **b** and with one gluon emission diagrams **c**, **d**)

$$\begin{aligned}&\langle V(p',\varepsilon)|\bar{q}\gamma_\mu\gamma_5 b|B_c(p)\rangle\\&\quad=-\mathrm{i}\bigg[\varepsilon^*_\mu(m_{B_c}+m_V)A_1(q^2)-(\varepsilon^*q)\mathcal{P}_\mu\frac{A_2(q^2)}{m_{B_c}+m_V}\\&\qquad-(\varepsilon^*q)\frac{2m_V}{q^2}\big[A_3(q^2)-A_0(q^2)\big]q_\mu\bigg],\end{aligned}\tag{2.4}$$

$$\begin{aligned}&\langle AV(p',\varepsilon)|\bar{q}\gamma_\mu\gamma_5 b|B_c(p)\rangle\\&\quad=-\epsilon_{\mu\nu\alpha\beta}\varepsilon^{*\nu}p^\alpha p'^\beta\frac{2V'(q^2)}{m_{B_c}+m_{AV}},\end{aligned}\tag{2.5}$$

$$\begin{aligned}&\langle AV(p',\varepsilon)|\bar{q}\gamma_\mu b|B_c(p)\rangle\\&\quad=-\mathrm{i}\bigg[\varepsilon^*_\mu(m_{B_c}+m_{AV})A'_1(q^2)-(\varepsilon^*q)\mathcal{P}_\mu\frac{A'_2(q^2)}{m_{B_c}+m_{AV}}\\&\qquad-(\varepsilon^*q)\frac{2m_{AV}}{q^2}\big[A'_3(q^2)-A'_0(q^2)\big]q_\mu\bigg],\end{aligned}\tag{2.6}$$

where $\mathcal{P}_\mu=(p+p')_\mu$, $q_\mu=(p-p')_\mu$ and ε^* is the polarization vector of the X mesons. To guarantee the finiteness of the results at $q^2=0$, we should have $A_3(0)(A'_3(0))=A_0(0)(A'_0(0))$. The form factor $A_3(q^2)(A'_3(q^2))$ can be written as a linear combination of $A_1(A'_1)$ and $A_2(A'_2)$ in the following way:

$$\begin{aligned}&A_3(q^2)(A'_3(q^2))\\&\quad=\frac{m_{B_c}+m_V(m_{AV})}{2m_V(m_{AV})}A_1(q^2)(A'_1(q^2))\\&\qquad-\frac{m_{B_c}-m_V(m_{AV})}{2m_V(m_{AV})}A_2(q^2)(A'_2(q^2)).\end{aligned}\tag{2.7}$$

Therefore, we need to calculate the form factors $V(V')$, $A_1(A'_1)$ and $A_2(A'_2)$. In order to obtain the sum rule expressions for these form factors, we consider the following three-point correlation function:

$$\begin{aligned}\Pi^{v;a}_{\mu\nu}&=\mathrm{i}^2\int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{e}^{-\mathrm{i}px}\mathrm{e}^{\mathrm{i}p'y}\\&\quad\times\langle 0|\mathcal{T}\{J_{X\nu}(y)J_\mu(0)^{v;a}J^\dagger_{B_c}(x)\}|0\rangle,\end{aligned}\tag{2.8}$$

where $J_{V\nu}(y)=\bar{c}\gamma_\nu q$, $J_{AV\nu}(y)=\bar{c}\gamma_5\gamma_\nu q$ $(q=s,d)$ and $J_{B_c}(x)=\mathrm{i}\bar{c}\gamma_5 b$ are the interpolating current of the V, AV and B_c mesons, respectively. $J^v_\mu=\bar{q}\gamma_\mu b$ and $J^a_\mu=\bar{q}\gamma_\mu\gamma_5 b$ are the vector and axial vector part of the transition current.

From the general philosophy of the QCD sum rules, we calculate the above correlation function in two languages. First, in hadron language, the results of the correlator give us the phenomenological or physical part; and the QCD or theoretical part of this correlator are obtained in the quark gluon languages. The sum rules for the form factors can be obtained by equating the coefficient of the corresponding structure from these two parts and applying a double Borel transformation with respect to the momenta of the initial and final meson states to eliminate the contributions coming from the higher states and continuum.

To calculate the phenomenological part of the correlator given in (2.8), two complete sets of intermediate states with the same quantum numbers as the currents J_V and J_{B_c} are inserted, respectively. As a result of this procedure, we get the following representation of the above-mentioned correlator:

$$\begin{aligned}&\Pi^{v;a}_{\mu\nu}(p^2,p'^2,q^2)\\&\quad=\frac{\langle 0|J_{X\nu}|X(p',\varepsilon)\rangle\langle X(p',\varepsilon)|J^{v;a}_\mu|B_c(p)\rangle\langle B_c(p)|J^\dagger_{B_c}|0\rangle}{(p'^2-m_X^2)(p^2-m_{B_c}^2)}\\&\qquad+\cdots,\end{aligned}\tag{2.9}$$

where $\cdots$ represent the contributions coming from the higher states and continuum. The matrix elements in (2.9) are defined in the standard way by

$$\begin{aligned}&\langle 0|J^\nu_X|X(p')\rangle=f_Xm_X\varepsilon^\nu,\\&\langle 0|J_{B_c}|B_c(p)\rangle=\mathrm{i}\frac{f_{B_c}m^2_{B_c}}{m_b+m_c},\end{aligned}\tag{2.10}$$

where f_X and f_{B_c} are the leptonic decay constants of X and B_c mesons, respectively. Using (2.3–2.6) and (2.10) in (2.9) and performing a summation over the polarization of X meson, we get the following result for the physical part:

$$\begin{aligned}&\Pi^{(v;a)}_{\nu\mu}(p^2,p'^2,q^2)\\&\quad=\frac{f_{B_c}f_{V(AV)}m^2_{B_c}m_{V(AV)}}{(m_b+m_c)(p_1^2-m^2_{B_c})(p_2^2-m^2_{V(AV)})}\\&\qquad\times\bigg\{\mathrm{i}\epsilon_{\nu\mu\alpha\beta}p^\alpha p'^\beta\frac{2V(V')}{m_{B_c}+m_{V(AV)}}\\&\qquad-(m_{B_c}+m_{V(AV)})\\&\qquad\times\bigg(-g_{\mu\nu}+\frac{(\mathcal{P}-q)_\mu(\mathcal{P}-q)_\nu}{4m^2_{V(AV)}}\bigg)A_1(A'_1)\\&\quad=\frac{1}{m_{B_c}+m_{V(AV)}}\mathcal{P}_\mu\bigg(-q_\nu+\frac{p'q(\mathcal{P}-q)_\nu}{2m^2_{V(AV)}}\bigg)A_2(A'_2)\\&\quad=\frac{2m_{V(AV)}}{q^2}q_\mu\bigg(-q_\nu+\frac{p'q(\mathcal{P}-q)_\nu}{2m^2_{V(AV)}}\bigg)\\&\qquad\times\big(A_3(A'_3)-A_0(A'_0)\big)\bigg\}.\end{aligned}\tag{2.11}$$

The coefficients of the Lorentz structures $i\epsilon_{\nu\mu\alpha\beta}p^\alpha p'^\beta$, $g_{\mu\nu}$ and $\mathcal{P}_\mu q_\nu$, give the expressions for the form factors $V(V')$, $A_1(A'_1)$ and $A_2(A'_2)$, respectively. The correlation function can be written in terms of the Lorentz structures in the following form:

$$\Pi^{(v;a)}_{\nu\mu}(p^2, p'^2, q^2) = \Pi_V \epsilon_{\nu\mu\alpha\beta} p^\alpha p'^\beta + \Pi_{A_1} g_{\mu\nu} + \Pi_{A_2} \mathcal{P}_\nu q_\mu + \cdots . \quad (2.12)$$

To calculate the QCD side of correlation function, on the other hand, we evaluate the three-point correlator with the help of the operator product expansion (OPE) in the deep Euclidean region $p^2 \ll (m_b + m_c)^2$, $p'^2 \ll (m_b^2 + m_q^2)$. To this aim, we write each $\Pi_{i(i')}$ [$i(i')$ stands for $V(V')$, $A_1(A'_1)$ and $A_2(A'_2)$] function in terms of the perturbative and nonperturbative parts as

$$\Pi_{i(i')}(p^2, p'^2, q^2) = \Pi^{\text{per}}_{i(i')}(p_1^2, p_2^2, q^2) + \Pi^{\text{non-per}}_{i(i')}(p^2, p'^2, q^2), \quad (2.13)$$

where $\langle \bar{q}q \rangle$ and $\langle G^2 \rangle$ denotes the light quark and two gluon condensates, respectively. For the perturbative part, we calculate the bare loop diagram (Fig. 1a); however, diagrams b, c, d in Fig. 1 correspond to the light quark condensates contributing to the correlation function. In principle, the light quark condensate diagrams give contributions to the correlation function, but applying double Borel transformations kills their contributions, so, as a first nonperturbative correction, we consider the gluon condensate diagrams (see Fig. 2a, b, c, d, e, f).

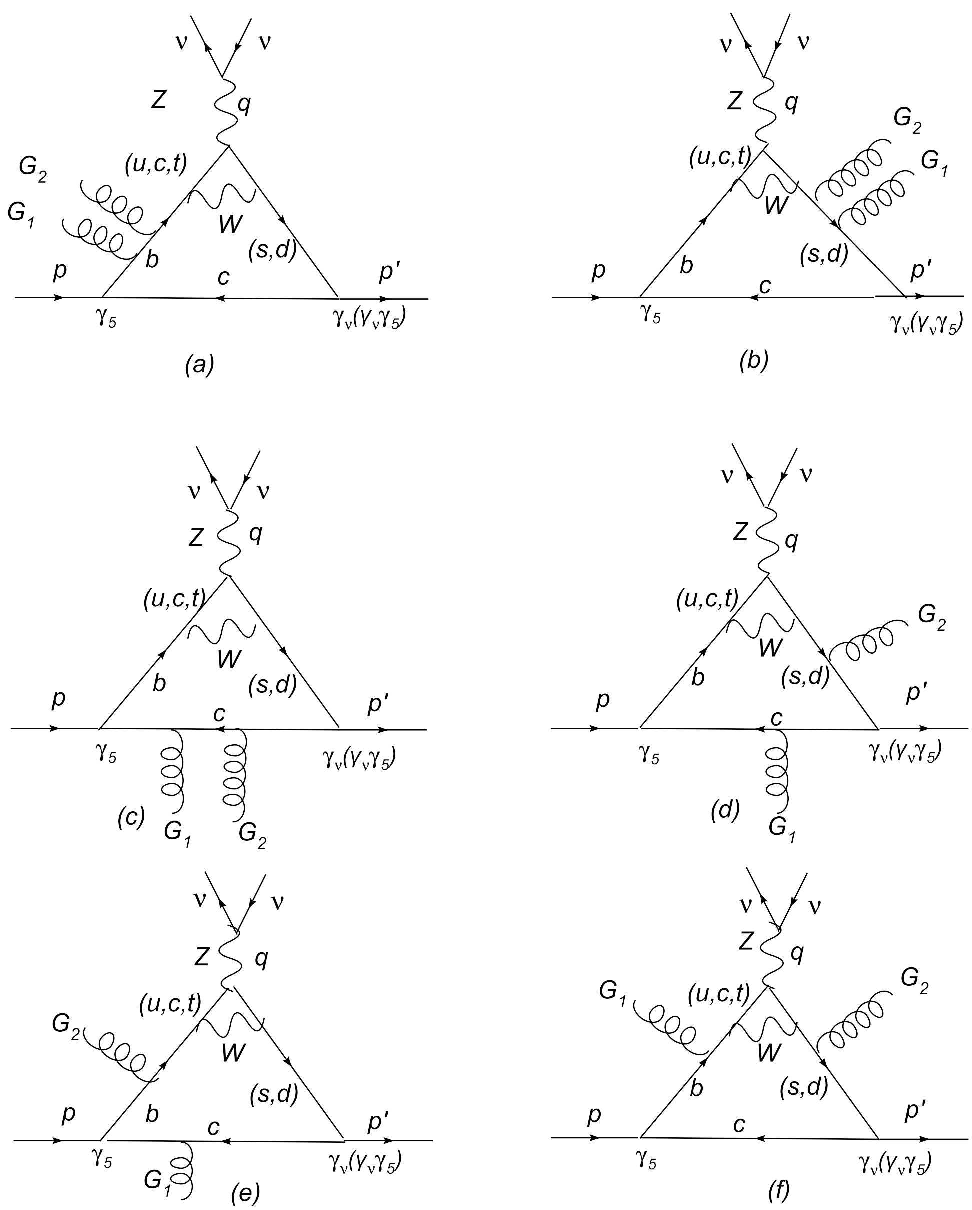

Fig. 2 Gluon condensate contributions to $B_c \to X\nu\bar{\nu}$ transitions

Using the double dispersion representation, the bare loop contribution is written as

$$\Pi_{i(i')}^{\text{per}} = -\frac{1}{(2\pi)^2}\int ds'\int ds\frac{\rho_{i(i')}^{\text{per}}(s,s',Q^2)}{(s-p^2)(s'-p'^2)} + \text{subtraction terms}, \tag{2.14}$$

where $Q^2=-q^2$. The spectral densities $\rho_{i(i')}^{\text{per}}(s,s',Q^2)$ are calculated with the help of the Cutkovsky rule, i.e., we replace the propagators with Dirac-delta functions:

$$\frac{1}{p^2-m^2}\to -2i\pi\delta\big(p^2-m^2\big), \tag{2.15}$$

implying that all quarks are real and the integration region in (2.14) is obtained by requiring that the arguments of three delta vanish, simultaneously. This condition leads to the following inequality:

$$-1\le \frac{2ss'+(s+s'+Q^2)(m_b^2-m_c^2-s)+2s(m_c^2-m_q^2)}{\lambda^{1/2}(s,s',-Q^2)\lambda^{1/2}(m_b^2,m_c^2,s)} \le +1, \tag{2.16}$$

where $\lambda(a,b,c)=a^2+b^2+c^2-2ab-2ac-2bc$. From this inequality, one can express s in terms of s', i.e. $f(s')$ in the s–s' plane.

Straightforward calculations lead to the following expressions for the spectral densities:

$$\rho_V\big(s,s',Q^2\big) = N_c I_0\big(s,s',-Q^2\big)\big[-4m_c+4(m_b-m_c)E_1 + 4(m_q-m_c)E_2\big], \tag{2.17}$$

$$\rho_{A_1}\big(s,s',Q^2\big) = N_c I_0\big(s,s',-Q^2\big)\big[8(m_c-m_b)D_1-4m_bm_cm_q + 4(m_q+m_b-m_c)m_c^2-2(m_q-m_c)\Delta - 2(m_b-m_c)\Delta'-2m_cu\big], \tag{2.18}$$

$$\rho_{A_2}\big(s,s',Q^2\big) = 2N_c I_0\big(s,s',-Q^2\big)\big[E_2m_b+D_3(m_b-m_c) + (E_1-E_2)m_c+D_2(m_c-m_b)-E_2m_q\big], \tag{2.19}$$

$$\rho_{V'}\big(s,s',Q^2\big) = -N_c I_0\big(s,s',-Q^2\big)\big[4m_c+4E_1(-m_b+m_c) + 4E_2(m_c+m_s)\big], \tag{2.20}$$

$$\rho_{A_1'}\big(s,s',Q^2\big) = -2N_c I_0\big(s,s',-Q^2\big)\big[4D_1(m_b-m_c)+\Delta'(m_b-m_c) - \Delta(m_c+m_s)+2{m_c}^2(m_c+m_s-m_b) - m_c(2m_bm_s-u)\big], \tag{2.21}$$

$$\rho_{A_2'}\big(s,s',Q^2\big) = -N_c I_0\big(s,s',-Q^2\big)\big[2D_2\,(m_b-m_c) + 2D_3\,(-m_b+m_c) + 2E_2(m_c-m_b-m_s)-2E_1m_c\big], \tag{2.22}$$

where $u=s+s'+Q^2$, $\Delta=s+m_c^2-m_b^2$, $\Delta'=s'+m_c^2-m_q^2$ and $N_c=3$ is the number of colors. The functions E_1, E_2, D_1, D_2, D_3 and I_0 are defined by

$$\begin{aligned}
I_0\big(s,s',-Q^2\big) &= \frac{1}{4\lambda^{1/2}(s,s',-Q^2)},\\
\lambda\big(s,s',-Q^2\big) &= s^2+s'^2+Q^4+2sQ^2+2s'Q^2-2ss',\\
E_1 &= \frac{1}{\lambda(s,s',-Q^2)}[2s'\Delta-\Delta'u],\\
E_2 &= \frac{1}{\lambda(s,s',-Q^2)}[2s\Delta'-\Delta u],\\
D_1 &= \frac{1}{2\lambda(s,s',-Q^2)} \times\big[\Delta'^2s+\Delta^2s'-4m_c^2ss'-\Delta\Delta'u+m_c^2u^2\big],\\
D_2 &= \frac{1}{\lambda^2(s,s',-Q^2)} \times\big[2\Delta'^2ss'+6\Delta^2s'^2-8m_c^2ss'^2 - 6\Delta\Delta's'u+\Delta'^2u^2+2m_c^2s'u^2\big],\\
D_3 &= \frac{1}{\lambda^2(s,s',-Q^2)} \times\big[2\Delta^2ss'+6\Delta'^2s^2-8m_c^2s's^2 - 6\Delta\Delta'su+\Delta^2u^2+2m_c^2su^2\big].
\end{aligned} \tag{2.23}$$

The next step is to calculate the gluon condensate contributions to the correlation function (diagrams in Fig. 2). In this section, we proceed from the definition of the integrals appearing in the evaluation of the gluon condensates contribution in the same way as in Refs. [13, 20]. The diagrams are calculated in the Fock–Schwinger fixed-point gauge [21–23]:

$$x^\mu G_\mu^a = 0, \tag{2.24}$$

where G_μ^a is the gluon field. In calculating the diagrams, the following type of integrals appear:

$$I_0[a,b,c] = \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2-m_b^2]^a[(p+k)^2-m_c^2]^b[(p'+k)^2-m_q^2]^c},$$
$$I_\mu[a,b,c] = \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{[k^2-m_b^2]^a[(p+k)^2-m_c^2]^b[(p'+k)^2-m_q^2]^c},$$
$$I_{\mu\nu}[a,b,c] = \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{[k^2-m_b^2]^a[(p+k)^2-m_c^2]^b[(p'+k)^2-m_q^2]^c}. \quad (2.25)$$

Here, k is the momentum of the spectator quark c. In the Schwinger representation for the propagators, i.e.,

$$\frac{1}{(p^2+m^2)^n} = \frac{1}{\Gamma(n)} \int_0^\infty d\alpha\, \alpha^{n-1} e^{-\alpha(p^2+m^2)}, \quad (2.26)$$

the integrals take a form suitable to application of the Borel transformations:

$$\mathcal{B}_{\hat{p}^2}(M^2) e^{-\alpha p^2} = \delta(1/M^2-\alpha). \quad (2.27)$$

Performing all integrals and applying double Borel transformations with respect to p^2 and p'^2, the Borel transformed form of the integrals are obtained as

$$\hat{I}_0(a,b,c) = \frac{(-1)^{a+b+c}}{16\pi^2\Gamma(a)\Gamma(b)\Gamma(c)} (M_1^2)^{2-a-b} (M_2^2)^{2-a-c} \times \mathcal{U}_0(a+b+c-4, 1-c-b),$$
$$\hat{I}_\mu(a,b,c) = \frac{1}{2}[\hat{I}_1(a,b,c)+\hat{I}_2(a,b,c)]\mathcal{P}_\mu + \frac{1}{2}[\hat{I}_1(a,b,c)-\hat{I}_2(a,b,c)]q_\mu, \quad (2.28)$$
$$\hat{I}_{\mu\nu}(a,b,c) = \hat{I}_6(a,b,c)g_{\mu\nu} + \frac{1}{4}(2\hat{I}_4+\hat{I}_3+\hat{I}_5)\mathcal{P}_\mu\mathcal{P}_\nu + \frac{1}{4}(-\hat{I}_5+\hat{I}_3)\mathcal{P}_\mu q_\nu + r\frac{1}{4}(-\hat{I}_5+\hat{I}_3)\mathcal{P}_\nu q_\mu + \frac{1}{4}(-2\hat{I}_4+\hat{I}_3+\hat{I}_5)q_\mu q_\nu,$$

where

$$\hat{I}_1(a,b,c) = i\frac{(-1)^{a+b+c+1}}{16\pi^2\Gamma(a)\Gamma(b)\Gamma(c)} (M_1^2)^{2-a-b} (M_2^2)^{3-a-c} \times \mathcal{U}_0(a+b+c-5, 1-c-b),$$
$$\hat{I}_2(a,b,c) = i\frac{(-1)^{a+b+c+1}}{16\pi^2\Gamma(a)\Gamma(b)\Gamma(c)} (M_1^2)^{3-a-b} (M_2^2)^{2-a-c} \times \mathcal{U}_0(a+b+c-5, 1-c-b),$$
$$\hat{I}_3(a,b,c) = i\frac{(-1)^{a+b+c}}{32\pi^2\Gamma(a)\Gamma(b)\Gamma(c)} (M_1^2)^{2-a-b} (M_2^2)^{4-a-c} \times \mathcal{U}_0(a+b+c-6, 1-c-b), \quad (2.29)$$
$$\hat{I}_4(a,b,c) = i\frac{(-1)^{a+b+c}}{16\pi^2\Gamma(a)\Gamma(b)\Gamma(c)} (M_1^2)^{3-a-b} (M_2^2)^{3-a-c} \times \mathcal{U}_0(a+b+c-6, 1-c-b),$$
$$\hat{I}_5(a,b,c) = i\frac{(-1)^{a+b+c}}{16\pi^2\Gamma(a)\Gamma(b)\Gamma(c)} (M_1^2)^{4-a-b} (M_2^2)^{2-a-c} \times \mathcal{U}_0(a+b+c-6, 1-c-b),$$
$$\hat{I}_6(a,b,c) = i\frac{(-1)^{a+b+c+1}}{16\pi^2\Gamma(a)\Gamma(b)\Gamma(c)} (M_1^2)^{3-a-b} (M_2^2)^{3-a-c} \times \mathcal{U}_0(a+b+c-6, 2-c-b).$$

Here, the hat in (2.28) and (2.29) denotes the double Borel transformed form of the integrals. M_1^2 and M_2^2 are the Borel parameters in the s and s' channels, respectively, and the function $\mathcal{U}_0(\alpha,\beta)$ is defined in the following way:

$$\mathcal{U}_0(a,b) = \int_0^\infty dy\,(y+M_1^2+M_2^2)^a y^b \times \exp\left[-\frac{B_{-1}}{y} - B_0 - B_1 y\right],$$

where

$$B_{-1} = \frac{1}{M_1^2M_2^2}\left[m_q^2M_1^4 + m_b^2M_2^4 + M_2^2M_1^2(m_b^2+m_q^2+Q^2)\right],$$
$$B_0 = \frac{1}{M_1^2M_2^2}\left[(m_q^2+m_c^2)M_1^2 + M_2^2(m_b^2+m_c^2)\right], \quad (2.30)$$
$$B_1 = \frac{m_c^2}{M_1^2M_2^2}.$$

After lengthy calculations, the following results for the gluon condensate contributions are obtained:

$$\Pi_{i(i')}^{\langle G^2\rangle} = -i\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\frac{C_{i(i')}}{12}, \quad (2.31)$$

where the explicit expressions for $C_{i(i')}$ and are given in Appendix A.

Applying double Borel transformations with respect to p^2 ($p^2 \to M_1^2$) and p'^2 ($p'^2 \to M_2^2$) on the phenomenological as well as the perturbative parts of the correlation function and equating the physical and QCD sides of the correlator, the following sum rules for the form factors V, A_1 and A_2 are obtained:

$$V(V') = -\frac{(m_b+m_c)(m_{B_c}+m_X)}{8\pi^2 f_{B_c} m_{B_c}^2 f_X m_X} e^{m_{B_c}^2/M_1^2} e^{m_X^2/M_2^2} \times \left\{\int_{(m_c+m_q)^2}^{s_0'} ds' \int_{f_-(s')}^{\min(s_0, f_+(s'))} ds\right.$$

$$\times\, \rho_{V(V')}\big(s,s',Q^2\big)\mathrm{e}^{-s/M_1^2}\mathrm{e}^{-s'/M_2^2} - \mathrm{i}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\frac{C_{V(V')}}{12}\bigg\},$$

$$\begin{aligned} A_1(A_1') = &-\frac{(m_b+m_c)}{4\pi^2 f_{B_c} m_{B_c}^2 f_X m_X (m_{B_c}+m_X)} \\ &\times \mathrm{e}^{m_{B_c}^2/M_1^2}\mathrm{e}^{m_X^2/M_2^2} \\ &\times \bigg\{\int_{(m_c+m_q)^2}^{s_0'}\mathrm{d}s'\int_{f_-(s')}^{\min(s_0,f_+(s'))}\mathrm{d}s \\ &\times \rho_{A_1(A_1')}\big(s,s',Q^2\big)\mathrm{e}^{-s/M_1^2}\mathrm{e}^{-s'/M_2^2} \\ &- \mathrm{i}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\frac{C_{A_1(A_1')}}{12}\bigg\}, \\ A_2(A_2') = &\frac{m_X(m_b+m_c)(m_{B_c}+m_X)}{\pi^2 f_{B_c} m_{B_c}^2 f_X (m_{B_c}^2+3m_X^2+Q^2)} \\ &\times \mathrm{e}^{m_{B_c}^2/M_1^2}\mathrm{e}^{m_X^2/M_2^2} \\ &\times \bigg\{\int_{(m_c+m_q)^2}^{s_0'}\mathrm{d}s'\int_{f_-(s')}^{\min(s_0,f_+(s'))}\mathrm{d}s \\ &\times \rho_{A_2(A_2')}\big(s,s',Q^2\big)\mathrm{e}^{-s/M_1^2}\mathrm{e}^{-s'/M_2^2} \\ &- \mathrm{i}\left\langle\frac{\alpha_s}{\pi}G^2\right\rangle\frac{C_{A_2(A_2')}}{12}\bigg\}, \end{aligned} \tag{2.32}$$

where s_0 and s_0' are the continuum thresholds in the B_c and X channels, respectively, and the $f_\pm(s')$ in the lower and upper limits of the integral over s are obtained from the inequality (2.16) with respect to s, i.e., $s=f_\pm(s')$. By $\min(s_0, f_+(s'))$, for each value of the q^2 between s_0 and f_+, the smaller one is selected. In the above equation, in order to subtract the contributions of the higher states and the continuum the quark–hadron duality assumption is also used; i.e., it is assumed that

$$\rho^{\text{higher states}}(s,s') = \rho^{\text{OPE}}(s,s')\theta(s-s_0)\theta(s-s_0'). \tag{2.33}$$

On the physical side and in the perturbative part of the correlation function, we also use the following Borel transformations:

$$\begin{aligned} \mathcal{B}_{p^2}\left\{\frac{1}{m^2(s)-p^2}\right\} &= \mathrm{e}^{\frac{m^2(s)}{M_1^2}}, \\ \mathcal{B}_{p'^2}\left\{\frac{1}{m^2(s')-p'^2}\right\} &= \mathrm{e}^{\frac{m^2(s')}{M_2^2}}. \end{aligned} \tag{2.34}$$

At the end of this section, we would like to present the differential decay width of $B_c \to X\nu\bar\nu$ decays in terms of the form factors. Using the amplitude in (2.2), we obtain the following expressions for the differential decay width of these transitions:

$$\begin{aligned} &\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2}(B_c\to V\nu\bar\nu) \\ &= \frac{G_F^2\alpha^2}{2^{10}\pi^5}\big|V_{tq}V_{tb}^*\big|^2\lambda^{1/2}(1,r_V,t')m_{B_c}^3|C_{10}|^2 \\ &\times\bigg(8\lambda(1,r_V,t')t'\frac{V^2}{(1+\sqrt{r_V})^2} \\ &+\frac{1}{r_V}\bigg[\lambda(1,r_V,t')^2\frac{A_2^2}{(1+\sqrt{r_V})^2} \\ &+(1+\sqrt{r_V})^2\big(\lambda(1,r_V,t')+12r_Vt'\big)A_1^2 \\ &-2\lambda(1,r_V,t')(1-r_V-t')Re(A_1A_2)\bigg]\bigg), \end{aligned} \tag{2.35}$$

$$\begin{aligned} &\frac{\mathrm{d}\Gamma}{\mathrm{d}q^2}(B_c\to AV\nu\bar\nu) \\ &= \frac{G_F^2\alpha^2}{2^{10}\pi^5}\big|V_{tq}V_{tb}^*\big|^2\lambda^{1/2}(1,r_{V'},t')m_{B_c}^3|C_{10}|^2 \\ &\times\bigg(8\lambda(1,r_{V'},t')t'\frac{V'^2}{(1+\sqrt{r_{V'}})^2} \\ &+\frac{1}{r_{V'}}\bigg[\lambda(1,r_{V'},t')^2\frac{A_2'^2}{(1+\sqrt{r_{V'}})^2} \\ &+(1+\sqrt{r_V'})^2\big(\lambda(1,r_{V'},t')+12r_V't'\big)A_1'^2 \\ &-2\lambda(1,r_{V'},t')(1-r_{V'}-t')Re(A_1'A_2')\bigg]\bigg), \end{aligned} \tag{2.36}$$

where $\lambda(1,r_V,t')$ and $\lambda(1,r_{V'},t')$ are the usual triangle functions with

$$\lambda(a,b,c) = a^2+b^2+c^2-2ab-2ac-2bc, \tag{2.37}$$

and

$$r_V = \frac{m_V^2}{m_{B_c}^2}, \qquad r_V' = \frac{m_{AV}^2}{m_{B_c}^2}, \qquad t' = -\frac{Q^2}{m_{B_c}^2}. \tag{2.38}$$

The total decay widths are obtained from integration of (2.35) and (2.36) over q^2 in the interval $0<q^2<(m_{B_c}-m_{V(AV)})^2$.

3 Numerical analysis

The explicit expressions for the form factors V, A_1, A_2, V', A_1' and A_2' and $\frac{d\Gamma}{dq^2}(B_c\to X\nu\bar\nu)$ indicate that the main input parameters entering the expressions are the gluon condensate, the Wilson coefficient C_{10}, the elements of the CKM matrix V_{tb}, V_{ts} and V_{td}, the leptonic decay constants; f_{B_C},

f_{D^*}, $f_{D_s^*}$ and $f_{D_{s1}}$, the Borel parameters M_1^2 and M_2^2, as well as the continuum thresholds s_0 and s_0'. For the numerical values of the gluon condensate, the leptonic decay constants, the CKM matrix elements, the Wilson coefficient and the quark and meson masses, we have the following values: $\langle \frac{\alpha_s}{\pi} G^2 \rangle = 0.012$ GeV^4 [24], $C_{10} = -4.669$ [25, 26], $|V_{tb}| = 0.77^{+0.18}_{-0.24}$, $|V_{ts}| = (40.6 \pm 2.7) \times 10^{-3}$ $|V_{td}| = (7.4 \pm 0.8) \times 10^{-3}$ [27], $f_{D_s^*} = 266 \pm 32$ MeV [28], $f_{D^*} = 0.23 \pm 0.02$ GeV [29], $f_{B_c} = 350$ MeV [30–32], $f_{D_{s1}} = 0.225 \pm 0.025$ GeV [28], $m_c(\mu = m_c) = 1.275 \pm 0.015$ GeV, $m_s(1\ \text{GeV}) \simeq 142$ MeV [33], $m_b = (4.7 \pm 0.1)$ GeV [34], $m_d = (3\text{–}7)$ MeV, $m_{D_s^*} = 2.112$ GeV, $m_{D^*} = 2.010$ GeV, $m_{D_{s1}} = 2.460$ GeV and $m_{B_C} = 6.258$ GeV [35].

The expressions for the form factors contain also four auxiliary parameters: the Borel mass squares M_1^2 and M_2^2, and the continuum thresholds s_0 and s_0'. These are not physical quantities; hence the physical quantities, the form factors, must be independent of these auxiliary parameters. We should find the "working regions" of these parameters where the form factors are independent of them. The parameters s_0 and s_0', which are the continuum thresholds of the B_c and X mesons, respectively, are determined from the conditions that guarantee the sum rules to have the best stability in the allowed M_1^2 and M_2^2 regions. The values of the continuum thresholds calculated from the two-point QCD sum rules are taken to be $s_0 = 45$ GeV^2 and $s_0' = 8$ GeV^2 [11, 24, 28]. The working regions for M_1^2 and M_2^2 are determined by requiring that, on the one hand, the continuum and higher states contributions are effectively suppressed, and on the other hand, the gluon condensate contribution is small, which guarantees that the contributions of higher dimensional operators are small. Both conditions are satisfied in the regions $10 \le M_1^2 \le 25$ GeV^2 and $5 \le M_2^2 \le 15$ GeV^2.

The values of the form factors at $q^2 = 0$ are shown in Table 1.

For obtaining the q^2 dependent expressions of the form factors, we should consider a range of q^2 where the sum rules can reliably be calculated. Our sum rules for the form factors are truncated at (1.2–2) GeV below the perturbative cut. In order to extend our results to the full physical region, i.e., the regions $0 \le q^2 \le 18$ GeV^2, $0 \le q^2 \le 17.2$ GeV^2 and $0 \le q^2 \le 14.4$ GeV^2 for $B_c \to D^* \nu\bar{\nu}$, $B_c \to D_s^* \nu\bar{\nu}$ and $B_c \to D_{s1} \nu\bar{\nu}$, respectively, we look for a parameterization of the form factors in such a way that this parameterization coincides with the sum rule prediction. Our numerical calculations show that the best parameterization of the form factors with respect to $-Q^2$ is as follows:

$$f_i(-Q^2) = \frac{f_i(0)}{1 + \alpha \hat{q} + \beta \hat{q}^2}, \tag{3.1}$$

where $\hat{q} = -Q^2/m_{B_c}^2$. The values of the parameters $f_i(0)$, α and β are given in Tables 2, 3 and 4 for $B_c \to D^* \nu\bar{\nu}$, $B_c \to D_s^* \nu\bar{\nu}$ and $B_c \to D_{s1} \nu\bar{\nu}$, respectively.

Having reached the end of this section, we would like to present the value of the branching ratio of these decays. Taking into account the q^2 dependencies of the form factors, and performing an integration over q^2 in (2.35) and (2.36) in the whole physical region, and using the total lifetime of the B_c meson, $\tau \simeq 0.46$ ps [36], the branching ratio of the $B_c \to X \nu\bar{\nu}$ decays are obtained as presented in Table 5. This table also encompasses a comparison of our

Table 1 The values of the form factors at $q^2 = 0$, for $M_1^2 = 18$ GeV^2, $M_2^2 = 8$ GeV^2

	$B_c \to D^* \nu\bar{\nu}$	$B_c \to D_s^* \nu\bar{\nu}$	$B_c \to D_{s1}(2460) \nu\bar{\nu}$
$V(0)$	0.27 ± 0.016	0.29 ± 0.017	0.30 ± 0.017
$A_1(0)$	0.12 ± 0.012	0.15 ± 0.014	0.13 ± 0.012
$A_2(0)$	-0.018 ± 0.0016	-0.03 ± 0.002	-0.07 ± 0.005

Table 2 Parameters appearing in the form factors of the $B_c \to D^* \nu\bar{\nu}$ decay in a two-parameter fit, for $M_1^2 = 18$ GeV^2 and $M_2^2 = 8$ GeV^2

	$f(0)$	α	β
V	0.27	−7.76	22.83
A_1	0.12	−8.37	22.29
A_2	−0.018	−11.63	33.52

Table 3 Parameters appearing in the form factors of the $B_c \to D_s^* \nu\bar{\nu}$ decay in a two-parameter fit, for $M_1^2 = 18$ GeV^2 and $M_2^2 = 8$ GeV^2

	$f(0)$	α	β
V	0.29	−3.17	9.90
A_1	0.15	−3.60	8.13
A_2	−0.03	−3.29	10.85

Table 4 Parameters appearing in the form factors of $B_c \to D_{s1} \nu\bar{\nu}$ decay in a two-parameter fit, for $M_1^2 = 18$ GeV^2 and $M_2^2 = 8$ GeV^2

	$f(0)$	α	β
V	0.30	−1.30	4.35
A_1	0.13	−2.40	4.43
A_2	−0.07	−1.25	4.30

Table 5 Our results for the branching ratios and their comparisons with the prediction of the relativistic constituent quark model (RCQM) [17]

Decay	Our result	RCQM [17]
$Br(B_c \to D^* \nu\bar{\nu}) \times 10^{-8}$	5.23 ± 0.12	5.78
$Br(B_c \to D_s^* \nu\bar{\nu}) \times 10^{-6}$	1.34 ± 0.25	1.42
$Br(B_c \to D_{s1} \nu\bar{\nu}) \times 10^{-6}$	1.73 ± 0.10	–

results with the existing predictions of the relativistic constituent quark model (RCQM) [17].

From this table, we see good consistency between our results and that of the relativistic constituent quark model.

In summary, we investigated the rare $B_c \to X\nu\bar{\nu}$ transition, with X being axial vector particles, $AV(D_{s1})$, and vector particles, $V(D^*, D_s^*)$, in the framework of the three-point QCD sum rule approach. The q^2 dependent expressions for the form factors were calculated. The quark condensate contributions to the correlation function were zero, so we considered the gluon corrections to the correlation function as first nonperturbative contributions. Finally, we calculated the total decay widths and branching ratio of these decays and compared our results with the predictions of the quark model. Our results are in good agreement with the relativistic constituent quark model.

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Appendix A

In this appendix we give the explicit expressions of the coefficients of the gluon condensate which enter the sum rules for the form factors V (V'), A_1 (A'_1) and A_2 (A'_2), respectively:

$$
\begin{aligned}
C_V ={}& -20I_1(3,2,2){m_c}^5 - 20I_2(3,2,2){m_c}^5 - 20I_0(3,2,2){m_c}^5 + 20I_1(3,2,2){m_c}^4 m_b \\
&+ 20I_1(3,2,2){m_c}^3{m_b}^2 + 20I_0(3,2,2){m_c}^3{m_b}^2 + 20I_2(3,2,2){m_c}^3{m_b}^2 - 20I_1(3,2,2){m_c}^2{m_b}^3 \\
&- 40I_0(2,2,2){m_c}^3 - 20I_2(3,1,2){m_c}^3 + 40I_1^{[0,1]}(3,2,2){m_c}^3 - 20I_0(3,1,2){m_c}^3 \\
&+ 40I_2^{[0,1]}(3,2,2){m_c}^3 - 60I_2(4,1,1){m_c}^3 - 40I_1(2,2,2){m_c}^3 + 20I_2(3,2,1){m_c}^3 \\
&- 40I_2(2,2,2){m_c}^3 - 60I_0(4,1,1){m_c}^3 + 40I_0^{[0,1]}(3,2,2){m_c}^3 - 60I_1(4,1,1){m_c}^3 \\
&- 20I_2(3,2,1){m_c}^2 m_b + 60I_1(4,1,1){m_c}^2 m_b + 80I_0(2,3,1){m_c}^2 m_b + 20I_0(3,2,1){m_c}^2 m_b \\
&+ 40I_1(2,2,2){m_c}^2 m_b + 20I_1(3,2,1)mc^2 m_b - 40I_1^{[0,1]}(3,2,2){m_c}^2 m_b + 40I_1(2,3,1){m_c}^2 m_b \\
&+ 120I_0(1,4,1)m_c{m_b}^2 + 20I_2^{[0,1]}(3,2,2)m_c{m_b}^2 + 20I_0^{[0,1]}(3,2,2)m_c{m_b}^2 + 20I_1^{[0,1]}(3,2,2)m_c{m_b}^2 \\
&+ 120I_1(1,4,1)m_c{m_b}^2 - 40I_0(3,2,1)m_c{m_b}^2 - 40I_2(3,2,1)m_c{m_b}^2 - 60I_1(3,2,1)m_c{m_b}^2 \\
&- 120I_1(1,4,1){m_b}^3 - 40I_1(2,3,1){m_b}^3 + 40I_1(3,2,1){m_b}^3 + 20I_1(2,2,2){m_b}^3 \\
&- 20I_1^{[0,1]}(3,2,2){m_b}^3 + 40I_1^{[0,1]}(3,1,2)m_c + 40I_1^{[0,1]}(3,2,1)m_c + 40I_2^{[0,1]}(2,2,2)m_c \\
&+ 60I_2^{[0,1]}(3,1,2)m_c - 20I_2^{[0,2]}(3,2,2)m_c - 20I_1(3,1,1)m_c - 20I_1^{[0,2]}(3,2,2)m_c \\
&+ 40I_1^{[0,1]}(2,2,2)m_c + 40I_1^{[0,1]}(3,2,1)m_c - 20I_0(3,1,1)m_c + 20I_2(3,1,1)m_c \\
&- 60I_0(2,1,2)m_c - 40I_1(1,2,2)m_c + 20I_2^{[0,1]}(3,2,1)m_c - 20I_0^{[0,2]}(3,2,2)m_c \\
&- 60I_2(2,1,2)m_c + 60I_0^{[0,1]}(3,1,2)m_c - 40I_2(1,2,2)m_c + 40I_0(2,2,1)m_c \\
&+ 60I_2(2,2,1)m_c - 40I_1(2,1,2)m_c - 40I_0(1,2,2)m_c + 40I_1(2,2,1)m_c \\
&+ 40I_0^{[0,1]}(2,2,2)m_c - 40I_0(2,2,1)m_b - 40I_1^{[0,1]}(3,1,2)m_b + 80I_2^{[0,1]}(2,3,1)m_b \\
&- 100I_1(2,2,1)m_b + 60I_1(2,1,2)m_b - 40I_1^{[0,1]}(3,2,1)m_b + 40I_1(1,2,2)m_b \\
&+ 200I_0(1,3,1)m_b + 40I_1^{[0,1]}(2,3,1)m_b + 120I_2(1,3,1)m_b + 20I_1^{[0,2]}(3,2,2)m_b \\
&- 40I_2(2,2,1)m_b - 40I_1^{[0,1]}(2,2,2)m_b + 40I_1(1,3,1)m_b,
\end{aligned}
$$

$$
\begin{aligned}
C_{A_1} ={}& 10I_0(3,2,2){m_c}^6 m_b - 10I_0(3,2,2){m_c}^5{m_b}^2 - 10I_0(3,2,2){m_c}^4{m_b}^3 + 10I_0(3,2,2){m_c}^3{m_b}^4 \\
&+ 40I_6(3,2,2){m_c}^5 + 30I_0(2,2,2){m_c}^4 m_b + 20I_0(2,3,1){m_c}^4 m_b - 40I_6(3,2,2){m_c}^4 m_b
\end{aligned}
$$

$$
\begin{aligned}
&- 30I_0^{[0,1]}(3,2,2)m_c{}^4m_b + 10I_0(3,2,1)m_c{}^4m_b + 30I_0(4,1,1)m_c{}^4m_b - 40I_6(3,2,2)m_c{}^3m_b{}^2\\
&+ 20I_0^{[0,1]}(3,2,2)m_c{}^3m_b{}^2 - 10I_0(3,2,1)m_c{}^3m_b{}^2 - 30I_0(4,1,1)m_c{}^3m_b{}^2 - 20I_0(2,2,2)m_c{}^3m_b{}^2\\
&- 60I_0(1,4,1)m_c{}^2m_b{}^3 + 40I_6(3,2,2)m_c{}^2m_b{}^3 + 20I_0(3,2,1)m_c{}^2m_b{}^3 - 20I_0(2,3,1)m_c{}^2m_b{}^3\\
&- 20I_0(3,2,1)m_cm_b{}^4 + 10I_0^{[0,1]}(3,2,2)m_cm_b{}^4 + 60I_0(1,4,1)m_cm_b{}^4 + 80I_6(3,2,1)m_c{}^3\\
&- 10I_0(3,1,1)m_c{}^3 + 40I_6(3,1,2)m_c{}^3 + 120I_6(4,1,1)m_c{}^3 - 80I_6^{[0,1]}(3,2,2)mc^3\\
&+ 80I_6(2,2,2)m_c{}^3 + 30I_0(2,1,2)m_c{}^2m_b + 20I_0(3,1,1)m_c{}^2m_b - 30I_0(2,2,1)m_c{}^2m_b\\
&- 20I_0^{[0,1]}(3,1,2)m_c{}^2m_b - 60I_0^{[0,1]}(2,2,2)m_c{}^2m_b - 40I_0^{[0,1]}(3,2,1)m_c{}^2m_b - 120I_6(4,1,1)m_c{}^2m_b\\
&- 30I_0^{[0,1]}(4,1,1)m_c{}^2m_b - 80I_6(3,2,1)m_c{}^2m_b - 40I_0^{[0,1]}(2,3,1)m_c{}^2m_b + 80I_6^{[0,1]}(3,2,2)m_c{}^2m_b\\
&- 40I_6(3,1,2)m_c{}^2m_b + 40I_0(1,2,2)m_c{}^2m_b - 20I_0(1,3,1)m_c{}^2m_b - 40I_6(2,2,2)m_c{}^2m_b\\
&+ 30I_0^{[o,2]}(3,2,2)m_c{}^2m_b + 80I_6(2,3,1)m_c{}^2m_b - 10I_0(3,1,1)m_cm_b{}^2 + 40I_6(2,2,2)m_cm_b{}^2\\
&- 30I_0(2,1,2)m_cm_b{}^2 - 40I_6^{[0,1]}(3,2,2)m_cm_b{}^2 - 240I_6(1,4,1)m_cm_b{}^2 + 20I_0(2,2,1)m_cm_b{}^2\\
&+ 60I_0(1,3,1)m_cm_b{}^2 + 10I_0^{[0,1]}(3,2,1)m_cm_b{}^2 + 40I_6(3,2,1)m_cm_b{}^2 + 20I_0^{[0,1]}(2,2,2)m_cm_b{}^2\\
&- 20I_0(1,2,2)m_cm_b{}^2 + 40I_0^{[0,1]}(3,1,2)m_cm_b{}^2 - 10I_0^{[0,2]}(3,2,2)m_cm_b{}^2 - 30I_0^{[0,1]}(3,2,1)m_b{}^3\\
&+ 40I_0(1,3,1)m_b{}^3 - 80I_6(2,3,1)m_b{}^3 + 10I_0^{[0,2]}(3,2,2)m_b{}^3 - 10I_0(2,2,1)m_b{}^3\\
&+ 40I_6^{[0,1]}(3,2,2)m_b{}^3 + 10I_0(1,2,2)m_b{}^3 + 240I_6(1,4,1)m_b{}^3 - 20I_0^{[0,1]}(2,2,2)mb^3\\
&- 40I_6(2,2,2)m_b{}^3 - 40I_6(3,2,1)m_b{}^3 + 60I_0^{[0,1]}(1,4,1)m_b{}^3 - 20I_0^{[0,1]}(2,3,1)m_b{}^3\\
&- 80I_6^{[0,1]}(2,2,2)m_c - 40I_6(2,1,2)m_c + 40I_6^{[0,1]}(3,2,2)m_c + 10I_0(2,1,1)m_c\\
&- 10I_0(1,1,2)m_c - 10I_0^{[0,1]}(3,1,1)m_c - 40I_6(2,2,1)m_c - 80I_6^{[0,1]}(3,2,1)m_c\\
&- 80I_6(3,1,1)m_c - 10I_0(1,2,1)m_c - 40I_6^{[0,1]}(3,1,2)m_c - 20I_0^{[0,1]}(3,1,1)m_b\\
&- 50I_0^{[0,1]}(2,1,2)m_b + 80I_6^{[0,1]}(3,1,2)m_b + 20I_0^{[0,1]}(3,1,2)m_b + 40I_6^{[0,1]}(2,2,2)m_b\\
&+ 30I_0(1,1,2)m_b - 40I_0^{[0,1]}(1,2,2)m_b + 80I_6^{[0,1]}(2,3,1)m_b + 20I_0^{[0,2]}(2,3,1)m_b\\
&+ 80I_6(3,1,1)m_b + 30I_0^{[0,1]}(2,2,2)m_b + 20I_0^{[0,1]}(1,3,1)m_b + 80I_6(1,3,1)m_b\\
&+ 40I_6(2,1,2)m_b + 30I_0^{[0,2]}(3,2,1)m_b + 40I_6^{[0,1]}(3,2,1)m_b - 30I_0^{[0,1]}(2,2,1)m_b\\
&- 40I_6^{[0,2]}(3,2,2)m_b + 30I_0(1,2,1)m_b + 80I_6(2,2,1)m_b - 30I_0(2,1,1)m_b + 40I_6(1,2,2)m_b,
\end{aligned}
$$

$$
\begin{aligned}
C_{A_2} = {}& 10I_3(4,1,2)m_c{}^5 + 10I_2(3,2,2)m_c{}^5 + 20I_2(4,2,1)m_c{}^5 + 10I_2(4,1,2)m_c{}^5\\
&- 10I_5(4,1,2)m_c{}^5 + 20I_1(4,2,1)m_c{}^5 + 10I_0(3,2,2)m_c{}^5 + 10I_1(4,1,2)m_c{}^5\\
&+ 10I_1(3,2,2)m_c{}^5 + 30I_5(3,3,1)m_c{}^5 - 15I_1(3,3,1)m_c{}^5 - 15I_2(3,3,1)m_c{}^5\\
&- 30I_5(4,2,1)m_c{}^5 - 30I_3(3,3,1)m_c{}^5 + 30I_3(4,2,1)m_c{}^5 - 10I_3(3,2,2)m_c{}^4m_b\\
&+ 10I_1(4,2,1)m_c{}^4m_b - 10I_5(4,1,2)m_c{}^4m_b - 30I_5(3,3,1)m_c{}^4m_b + 80I_5(2,4,1)m_c{}^4m_b\\
&- 10I_2(2,4,1)m_c{}^4m_b - 10I_1(2,4,1)m_c{}^4m_b + 10I_3(4,1,2)m_c{}^4m_b + 10I_2(4,2,1)m_c{}^4m_b\\
&+ 30I_3(3,3,1)m_c{}^4m_b - 80I_3(2,4,1)m_c{}^4m_b + 10I_5(3,2,2)m_c{}^4m_b + 40I_3(2,4,1)m_c{}^2m_b{}^3\\
&- 40I_5(2,4,1)m_c{}^2m_b{}^3 - 20I_3(2,4,1)m_b{}^5 + 20I_5(2,4,1)m_b{}^5 + 20I_1(2,2,2)m_c{}^3\\
&- 30I_2(4,1,1)m_c{}^3 + 10I_2^{[0,1]}(3,2,2)m_c{}^3 + 20I_0(2,2,2)m_c{}^3 + 20I_2(2,2,2)m_c{}^3
\end{aligned}
$$

$$
\begin{aligned}
&- 15I_0(3,1,2){m_c}^3 - 10I_1^{[0,1]}(3,2,2){m_c}^3 - 30I_1(4,1,1){m_c}^3 + 40I_1(3,2,1){m_c}^3\\
&+ 20I_2(3,1,2){m_c}^3 + 40I_2(3,2,1){m_c}^3 - 30I_1(2,3,1){m_c}^3 - 30I_2(2,3,1){m_c}^3\\
&+ 20I_1(3,1,2){m_c}^3 - 20I_1(1,4,1){m_c}^2m_b - 20I_1^{[0,1]}(2,4,1){m_c}^2m_b + 20I_2^{[0,1]}(2,4,1){m_c}^2m_b\\
&- 5I_0(2,2,2){m_c}^2m_b + 40I_5^{[0,1]}(2,4,1){m_c}^2m_b + 20I_2(3,2,1){m_c}^2m_b + 10I_0(3,1,2){m_c}^2m_b\\
&- 40I_3^{[0,1]}(2,4,1){m_c}^2m_b - 20I_2(1,4,1){m_c}^2m_b + 20I_1(3,2,1){m_c}^2m_b - 5I_1^{[0,1]}(3,2,2)m_c{m_b}^2\\
&- 30I_1(1,4,1)m_c{m_b}^2 + 5I_2^{[0,1]}(3,2,2)m_c{m_b}^2 - 10I_0(2,2,2)m_c{m_b}^2 - 30I_2(1,4,1)m_c{m_b}^2\\
&+ 5I_0(3,1,2)m_c{m_b}^2 - 40I_5^{[0,1]}(2,4,1){m_b}^3 + 40I_3^{[0,1]}(2,4,1){m_b}^3 - 5I_0(2,2,2){m_b}^3\\
&- 10I_1^{[0,1]}(2,2,2)m_c + 15I_1(2,1,2)m_c + 20I_2(2,2,1)m_c - 15I_2^{[0,1]}(2,3,1)m_c\\
&+ 15I_2(2,1,2)m_c + 5I_0(2,1,2)m_c + 5I_1(1,2,2)m_c - 5I_2^{[0,2]}(3,2,2)m_c\\
&+ 10I_2^{[0,1]}(3,2,1)m_c + 10I_0(1,2,2)m_c - 15I_2(3,1,1)m_c - 15I_2(1,3,1)m_c\\
&+ 5I_2(1,2,2)m_c + 5I_1^{[0,2]}(3,2,2)m_c - 5I_0^{[0,1]}(3,1,2)m_c + 15I_1^{[0,1]}(2,3,1)m_c\\
&- 15I_1(3,1,1)m_c + 20I_1(2,2,1)m_c - 15I_1(1,3,1)m_c + 10I_2^{[0,1]}(2,2,2)m_c\\
&+ 10I_2^{[0,1]}(3,1,2)m_c - 10I_1^{[0,1]}(3,2,1)m_c - 10I_1^{[0,1]}(3,1,2)m_c + 10I_2(2,2,1)m_b\\
&- 25I_0(2,1,2)m_b - 10I_2^{[0,1]}(1,4,1)m_b + 40I_5^{[0,1]}(1,4,1)m_b + 5I_2^{[0,1]}(2,2,2)m_b\\
&- 5I_0(1,2,2)m_b + 40I_3^{[0,1]}(2,3,1)m_b - 40I_5^{[0,1]}(2,3,1)m_b + 20I_5^{[0,2]}(2,4,1)m_b\\
&+ 10I_1^{[0,1]}(1,4,1)m_b - 5I_1^{[0,1]}(2,2,2)m_b + 10I_1(2,2,1)m_b - 40I_3^{[0,1]}(1,4,1)m_b\\
&- 20I_3^{[0,2]}(2,4,1)m_b + 5I_0^{[0,1]}(2,2,2)m_b,
\end{aligned}
$$

$$
\begin{aligned}
C_{V'} = {}&-20I_1(3,2,2){m_c}^5 - 20I_2(3,2,2){m_c}^5 - 20I_0(3,2,2){m_c}^5 + 20I_1(3,2,2){m_c}^4m_b\\
&+ 20I_2(3,2,2){m_c}^3{m_b}^2 + 20I_1(3,2,2){m_c}^3{m_b}^2 + 20I_0(3,2,2){m_c}^3{m_b}^2 - 20I_1(3,2,2){m_c}^2{m_b}^3\\
&- 40I_0(2,2,2){m_c}^3 + 40I_0^{[0,1]}(3,2,2){m_c}^3 - 40I_2(2,2,2){m_c}^3 - 40I_1(2,2,2){m_c}^3\\
&+ 40I_1^{[0,1]}(3,2,2){m_c}^3 - 60I_0(4,1,1){m_c}^3 + 40I_2^{[0,1]}(3,2,2){m_c}^3 - 60I_2(4,1,1){m_c}^3\\
&+ 20I_2(3,2,1){m_c}^3 - 20I_0(3,1,2){m_c}^3 - 60I_1(4,1,1){m_c}^3 + 60I_2(3,1,2){m_c}^3\\
&- 20I_2(3,2,1){m_c}^2m_b - 40I_1^{[0,1]}(3,2,2){m_c}^2m_b + 20I_1(3,2,1){m_c}^2m_b + 60I_1(4,1,1){m_c}^2m_b\\
&+ 40I_1(2,3,1){m_c}^2m_b + 80I_0(2,3,1){m_c}^2m_b + 20I_0(3,2,1){m_c}^2m_b + 40I_1(2,2,2){m_c}^2m_b\\
&- 60I_1(3,2,1)m_c{m_b}^2 + 20I_1^{[0,1]}(3,2,2)m_c{m_b}^2 + 20I_0^{[0,1]}(3,2,2)m_c{m_b}^2 + 120I_0(1,4,1)m_c{m_b}^2\\
&- 40I_0(3,2,1)m_c{m_b}^2 - 40I_2(3,2,1)m_c{m_b}^2 + 120I_2(1,4,1)m_c{m_b}^2 + 120I_1(1,4,1)m_c{m_b}^2\\
&+ 20I_2^{[0,1]}(3,2,2)m_c{m_b}^2 - 120I_1(1,4,1){m_b}^3 - 20I_1^{[0,1]}(3,2,2){m_b}^3 + 40I_1(3,2,1){m_b}^3\\
&+ 20I_1(2,2,2){m_b}^3 - 40I_1(2,3,1){m_b}^3 + 60I_0^{[0,1]}(3,1,2)m_c + 40I_0(2,2,1)m_c\\
&- 20I_0^{[0,2]}(3,2,2)m_c - 20I_1(3,1,1)m_c - 40I_1(1,2,2)m_c + 20I_2(2,1,2)m_c\\
&- 40I_1(2,1,2)m_c + 20I_2(3,1,1)m_c - 20I_0(3,1,1)m_c + 40I_2^{[0,1]}(2,2,2)m_c\\
&+ 40I_1(2,2,1)m_c + 40I_0^{[0,1]}(2,2,2)m_c - 60I_0(2,1,2)m_c - 20I_2^{[0,1]}(3,1,2)m_c\\
&- 20I_1^{[0,2]}(3,2,2)m_c + 40I_1^{[0,1]}(3,2,1)m_c - 40I_0(1,2,2m_c + 20I_2^{[0,1]}(3,2,1)m_c\\
&+ 60I_2(2,2,1)m_c + 40I_0^{[0,1]}(3,2,1)m_c + 40I_1^{[0,1]}(2,2,2)m_c - 20I_2^{[0,2]}(3,2,2)m_c
\end{aligned}
$$

$$+40I_1^{[0,1]}(3,1,2)m_c-40I_2(1,2,2)m_c+200I_0(1,3,1)m_b+80I_2^{[0,1]}(2,3,1)m_b$$
$$+40I_1(1,2,2)m_b+40I_1^{[0,1]}(2,3,1)m_b+60I_1(2,1,2)m_b-40I_1^{[0,1]}(3,1,2)m_b$$
$$-40I_0(2,2,1)m_b+120I_2(1,3,1)m_b-100I_1(2,2,1)m_b-40I_1^{[0,1]}(3,2,1)m_b$$
$$-40I_1^{[0,1]}(2,2,2)m_b+40I_1(1,3,1)m_b-40I_2(2,2,1)m_b+20I_1^{[0,2]}(3,2,2)m_b,$$

$$C_{A_1'}=-10I_0(3,2,2){m_c}^6m_b+10I_0(3,2,2){m_c}^5{m_b}^2+10I_0(3,2,2){m_c}^4{m_b}^3-10I_0(3,2,2){m_c}^3{m_b}^4$$
$$-40I_6(3,2,2){m_c}^5+10I_0(2,1,3){m_c}^4m_b+40I_6(3,2,2){m_c}^4m_b-10I_0(3,2,1){m_c}^4m_b$$
$$-30I_0(4,1,1){m_c}^4m_b-30I_0(2,2,2)mc^4m_b+30I_0^{[0,1]}(3,2,2){m_c}^4m_b-20I_0(2,3,1){m_c}^4m_b$$
$$+40I_6(3,2,2){m_c}^3{m_b}^2+30I_0(4,1,1){m_c}^3{m_b}^2-10I_0(2,1,3){m_c}^3{m_b}^2+20I_0(2,2,2){m_c}^3{m_b}^2$$
$$-20I_0^{[0,1]}(3,2,2){m_c}^3{m_b}^2+10I_0(3,2,1){m_c}^3{m_b}^2-40I_6(3,2,2){m_c}^2{m_b}^3+60I_0(1,4,1){m_c}^2{m_b}^3$$
$$-20I_0(3,2,1){m_c}^2{m_b}^3+20I_0(2,3,1){m_c}^2{m_b}^3-60I_0(1,4,1)m_c{m_b}^4-10I_0^{[0,1]}(3,2,2)m_c{m_b}^4$$
$$+20I_0(3,2,1)m_c{m_b}^4-40I_6(3,1,2){m_c}^3+80I_6^{[0,1]}(3,2,2){m_c}^3+10I_0(2,1,2){m_c}^3$$
$$-80I_6(2,2,2){m_c}^3+10I_0(3,1,1){m_c}^3-80I_6(3,2,1){m_c}^3-120I_6(4,1,1){m_c}^3$$
$$-30I_0^{[0,2]}(3,2,2){m_c}^2m_b-80I_6^{[0,1]}(3,2,2){m_c}^2m_b+40I_6(3,1,2){m_c}^2m_b+30I_0(2,2,1){m_c}^2m_b$$
$$+20I_0^{[0,1]}(3,1,2){m_c}^2m_b+80I_6(3,2,1){m_c}^2m_b+40I_0^{[0,1]}(3,2,1){m_c}^2m_b+30I_0^{[0,1]}(4,1,1){m_c}^2m_b$$
$$+120I_6(4,1,1){m_c}^2m_b-20I_0^{[0,1]}(2,1,3){m_c}^2m_b+40I_0^{[0,1]}(2,3,1){m_c}^2m_b-50I_0(2,1,2){m_c}^2m_b$$
$$+20I_0(1,1,3){m_c}^2m_b+20I_0(1,3,1)mc^2m_b-20I_0(3,1,1){m_c}^2m_b+60I_0^{[0,1]}(2,2,2){m_c}^2m_b$$
$$-40I_0(1,2,2){m_c}^2m_b-80I_6(2,3,1)mc^2m_b+40I_6(2,2,2){m_c}^2m_b-40I_6(3,2,1)m_c{m_b}^2$$
$$+240I_6(1,4,1)m_c{m_b}^2+20I_0(1,2,2)m_c{m_b}^2-40I_0^{[0,1]}(3,1,2)m_c{m_b}^2+10I_0^{[0,1]}(2,1,3)m_c{m_b}^2$$
$$-20I_0^{[0,1]}(2,2,2)m_c{m_b}^2-10I_0^{[0,1]}(3,2,1)m_c{m_b}^2+40I_6^{[0,1]}(3,2,2)m_c{m_b}^2-60I_0(1,3,1)m_c{m_b}^2$$
$$-10I_0(1,1,3)m_c{m_b}^2+10I_0(3,1,1)m_c{m_b}^2+10I_0^{[0,2]}(3,2,2)m_c{m_b}^2-20I_0(2,2,1)m_c{m_b}^2$$
$$+40I_0(2,1,2)m_c{m_b}^2-40I_6(2,2,2)m_c{m_b}^2-10I_0(1,2,2){m_b}^3-40I_0(1,3,1){m_b}^3$$
$$+10I_0(2,2,1){m_b}^3+80I_6(2,3,1){m_b}^3+40I_6(3,2,1){m_b}^3-240I_6(1,4,1){m_b}^3$$
$$+20I_0^{[0,1]}(2,3,1){m_b}^3+40I_6(2,2,2){m_b}^3-10I_0^{[0,1]}(3,2,2){m_b}^3+20I_0^{[0,1]}(2,2,2){m_b}^3$$
$$+30I_0^{[0,1]}(3,2,1){m_b}^3-60I_0^{[0,1]}(1,4,1){m_b}^3-40I_6^{[0,1]}(3,2,2){m_b}^3-80I_6(2,2,1)m_c$$
$$+20I_0(1,1,2)m_c-10I_0^{[0,1]}(2,1,2)m_c+10I_0^{[0,1]}(3,1,1)m_c+80I_6(3,1,1)m_c$$
$$+40I_6(2,1,2)m_c+40I_6^{[0,1]}(3,1,2)m_c-40I_6^{[0,2]}(3,2,2)m_c-20I_0(2,1,1)m_c$$
$$+80I_6^{[0,1]}(3,2,1)m_c+10I_0(1,2,1)m_c+80I_6^{[0,1]}(2,2,2)m_c-40I_6^{[0,1]}(2,2,2)m_b$$
$$-80I_6^{[0,1]}(3,1,2)m_b-20I_0^{[0,2]}(2,3,1)m_b+30I_0^{[0,1]}(2,1,2)m_b-30I_0(1,2,1)m_b$$
$$+40I_6^{[0,2]}(3,2,2)m_b+40I_6(2,2,1)m_b-40I_6(2,1,2)m_b+40I_0(2,1,1)m_b$$
$$+10I_0^{[0,2]}(2,1,3)m_b-20I_0^{[0,1]}(1,3,1)m_b+20I_0^{[0,1]}(3,1,1)m_b-20I_0^{[0,1]}(1,1,3)m_b$$
$$-30I_0^{[0,2]}(2,2,2)m_b+40I_0^{[0,1]}(1,2,2)m_b-80I_6(3,1,1)m_b+30I_0^{[0,1]}(2,2,1)m_b$$
$$-20I_0^{[0,2]}(3,1,2)m_b-40I_6(1,2,2)m_b-40I_6^{[0,1]}(3,2,1)m_b-80I_6^{[0,1]}(2,3,1)m_b$$
$$-80I_6(1,3,1)m_b-30I_0^{[0,2]}(3,2,1)m_b-30I_0(1,1,2)m_b,$$

$$
\begin{aligned}
C_{A_2'} = &-10I_1(3,2,2)m_c{}^5 - 10I_3(3,2,2)m_c{}^5 - 5I_0(3,2,2)m_c{}^5 + 10I_5(3,2,2)m_c{}^5 \\
&- 5I_2(3,2,2)m_c{}^4m_b + 5I_1(3,2,2)mc^4m_b - 10I_5(3,2,2)m_c{}^4m_b + 10I_3(3,2,2)m_c{}^4m_b \\
&+ 10I_3(3,2,2)m_c{}^3m_b{}^2 + 10I_1(3,2,2)m_c{}^3m_b{}^2 + 5I_0(3,2,2)m_c{}^3m_b{}^2 - 10I_5(3,2,2)m_c{}^3m_b{}^2 \\
&+ 10I_5(3,2,2)m_c{}^2m_b{}^3 - 10I_3(3,2,2)m_c{}^2m_b{}^3 - 5I_1(3,2,2)m_c{}^2mb^3 + 5I_2(3,2,2)m_c{}^2m_b{}^3 \\
&+ 30I_1^{[0,1]}(3,2,2)m_c{}^3 - 10I_0(2,2,2)m_c{}^3 + 10I_2(3,1,2)m_c{}^3 - 20I_1(3,1,2)m_c{}^3 \\
&- 5I_0(3,1,2)m_c{}^3 + 10I_0^{[0,1]}(3,2,2)m_c{}^3 - 25I_1(3,2,1)m_c{}^3 + 20I_3^{[0,1]}(3,2,2)m_c{}^3 \\
&- 15I_2(3,2,1)m_c{}^3 - 25I_0(3,2,1)m_c{}^3 - 20I_5^{[0,1]}(3,2,2)m_c{}^3 - 15I_0(4,1,1)m_c{}^3 \\
&- 10I_2^{[0,1]}(3,2,2)m_c{}^3 - 30I_1(4,1,1)m_c{}^3 - 20I_1(2,2,2)m_c{}^3 + 5I_1(2,2,2)m_c{}^2m_b \\
&- 10I_1(2,3,1)m_c{}^2m_b - 5I_2(2,2,2)mc^2m_b - 20I_0(2,3,1)m_c{}^2m_b + 20I_5^{[0,1]}(3,2,2)m_c{}^2m_b \\
&- 10I_1^{[0,1]}(3,2,2)m_c{}^2m_b - 20I_3^{[0,1]}(3,2,2)m_c{}^2m_b + 10I_2(2,3,1)m_c{}^2m_b + 10I_1(3,1,2)m_c{}^2m_b \\
&+ 15I_1(4,1,1)m_c{}^2m_b + 10I_2^{[0,1]}(3,2,2)m_c{}^2m_b + 5I_0(3,2,1)m_c{}^2m_b + 20I_1(3,2,1)m_c{}^2m_b \\
&- 15I_2(4,1,1)m_c{}^2m_b - 10I_2(3,1,2)mc^2m_b + 10I_2(3,2,1)m_c{}^2m_b + 10I_2(2,2,2)m_cm_b{}^2 \\
&- 20I_1(3,2,1)m_cm_b{}^2 + 30I_0(1,4,1)m_cm_b{}^2 - 10I_1(2,2,2)m_cm_b{}^2 + 10I_3^{[0,1]}(3,2,2)m_cm_b{}^2 \\
&+ 5I_0^{[0,1]}(3,2,2)m_cm_b{}^2 - 15I_0(3,2,1)m_cm_b{}^2 - 5I_2^{[0,1]}(3,2,2)m_cm_b{}^2 + 15I_1^{[0,1]}(3,2,2)m_cm_b{}^2 \\
&+ 60I_1(1,4,1)m_cm_b{}^2 - 5I_2(3,1,2)mcm_b{}^2 - 10I_5^{[0,1]}(3,2,2)m_cm_b{}^2 + 5I_1(3,1,2)m_cm_b{}^2 \\
&- 10I_2(3,2,1)m_b{}^3 + 5I_2^{[0,1]}(3,2,2)m_b{}^3 - 30I_1(1,4,1)m_b{}^3 + 10I_1(2,3,1)m_b{}^3 \\
&- 5I_1^{[0,1]}(3,2,2)m_b{}^3 + 10I_1(3,2,1)m_b{}^3 + 10I_5^{[0,1]}(3,2,2)m_b{}^3 - 10I_3^{[0,1]}(3,2,2)m_b{}^3 \\
&+ 30I_2(1,4,1)m_b{}^3 - 10I_2(2,3,1)m_b{}^3 + 20I_3^{[0,1]}(2,2,2)m_c - 20I_0(2,2,1)m_c \\
&- 5I_0^{[0,2]}(3,2,2)m_c - 20I_5^{[0,1]}(3,2,1)m_c - 15I_1^{[0,2]}(3,2,2)m_c + 10I_3^{[0,1]}(3,1,2)m_c \\
&- 10I_3^{[0,2]}(3,2,2)m_c + 5I_2(3,1,1)m_c - 10I_0(3,1,1)m_c + 10I_5^{[0,2]}(3,2,2)m_c \\
&+ 30I_1^{[0,1]}(3,1,2)m_c + 30I_1^{[0,1]}(2,2,2)m_c + 20I_3^{[0,1]}(3,2,1)m_c + 15I_0^{[0,1]}(3,2,1)m_c \\
&- 25I_1(2,2,1)m_c - 20I_5^{[0,1]}(2,2,2)m_c - 15I_0(2,1,2)m_c - 20I_1(2,1,2)m_c \\
&- 10I_5^{[0,1]}(3,1,2)m_c + 15I_0^{[0,1]}(3,1,2)m_c + 5I_2^{[0,2]}(3,2,2)m_c + 5I_1(3,1,1)m_c \\
&- 15I_2^{[0,1]}(3,2,1)m_c - 10I_2^{[0,1]}(2,2,2)m_c - 10I_2(2,1,2)m_c + 35I_1^{[0,1]}(3,2,1)m_c \\
&- 5I_2(2,2,1)m_c + 10I_0^{[0,1]}(2,2,2)m_c - 10I_1^{[0,1]}(2,3,1)m_b + 10I_0(1,3,1)m_b \\
&+ 5I_1^{[0,2]}(3,2,2)m_b - 10I_3^{[0,1]}(2,2,2)m_b + 10I_2^{[0,1]}(3,2,1)m_b + 15I_2(1,2,2)m_b \\
&- 20I_2(2,2,1)m_b + 10I_3^{[0,2]}(3,2,2)m_b - 15I_1(1,2,2)m_b + 20I_5^{[0,1]}(3,1,2)m_b \\
&- 5I_2^{[0,2]}(3,2,2)m_b + 20I_2(2,1,2)m_b + 10I_2^{[0,1]}(2,3,1)m_b - 20I_3^{[0,1]}(3,1,2)m_b \\
&- 20I_3^{[0,1]}(2,3,1)m_b + 10I_5^{[0,1]}(2,2,2)m_b + 20I_5^{[0,1]}(2,3,1)m_b + 10I_2^{[0,1]}(3,1,2)m_b \\
&- 10I_1^{[0,1]}(3,1,2)m_b + 10I_5^{[0,1]}(3,2,1)m_b - 10I_0(2,2,1)m_b - 10I_1^{[0,1]}(3,2,1)m_b,
\end{aligned}
$$

where

$$
\hat{I}_n^{[i,j]}(a,b,c) = \left(M_1^2\right)^i \left(M_2^2\right)^j \frac{d^i}{d(M_1^2)^i} \frac{d^j}{d(M_2^2)^j} \left[\left(M_1^2\right)^i \left(M_2^2\right)^j \hat{I}_n(a,b,c)\right].
$$

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